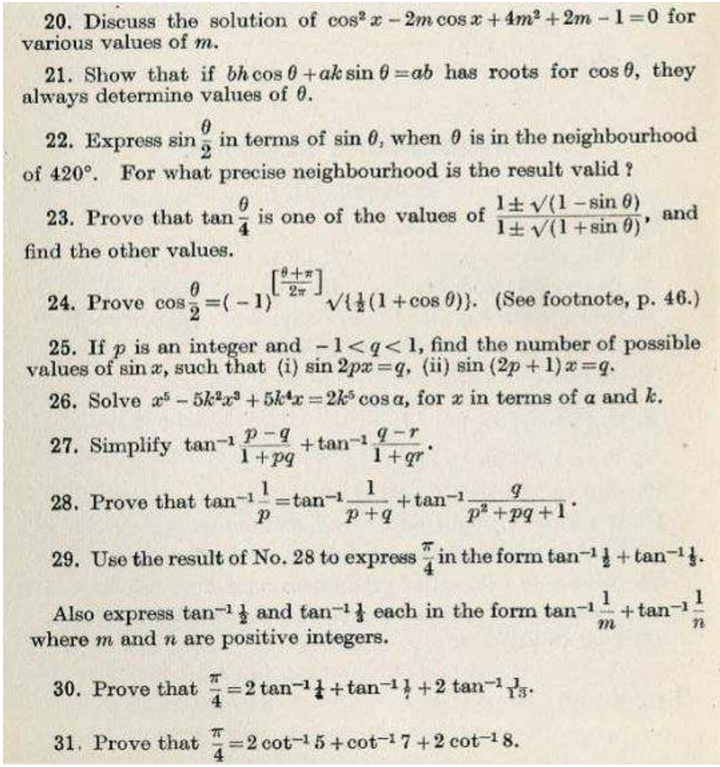
Trigonometry practice in a book published in the year 1930...



20. Discuss the solution of $\cos^2 x - 2m\cos x + 4m^2 + 2m - 1 = 0$ for various values of m.

21. Show that if $bh\cos\theta + ak\sin\theta = ab$ has roots for $\cos\theta$, they always determine values of θ.

22. Express $\sin\frac{\theta}{2}$ in terms of $\sin\theta$, when θ is in the neighbourhood of 420°. For what precise neighbourhood is the result valid ?

23. Prove that $\tan\frac{\theta}{4}$ is one of the values of $\frac{1\pm\sqrt{(1-\sin\theta)}}{1\pm\sqrt{(1+\sin\theta)}}$, and find the other values.

24. Prove $\cos\frac{\theta}{2} = (-1)^{\left[\frac{\theta+\pi}{2\pi}\right]}\sqrt{\{\frac{1}{2}(1+\cos\theta)\}}$. (See footnote, p. 46.)

25. If p is an integer and $-1 < q < 1$, find the number of possible values of $\sin x$, such that (i) $\sin 2px = q$, (ii) $\sin(2p+1)x = q$.

26. Solve $x^5 - 5k^2x^3 + 5k^4x = 2k^5\cos a$, for x in terms of a and k.

27. Simplify $\tan^{-1}\frac{p-q}{1+pq} + \tan^{-1}\frac{q-r}{1+qr}$.

28. Prove that $\tan^{-1}\frac{1}{p} = \tan^{-1}\frac{1}{p+q} + \tan^{-1}\frac{q}{p^2+pq+1}$.

29. Use the result of No. 28 to express $\frac{\pi}{4}$ in the form $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$.

Also express $\tan^{-1}\frac{1}{2}$ and $\tan^{-1}\frac{1}{3}$ each in the form $\tan^{-1}\frac{1}{m} + \tan^{-1}\frac{1}{n}$ where m and n are positive integers.

30. Prove that $\frac{\pi}{4} = 2\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{13}$.

31. Prove that $\frac{\pi}{4} = 2\cot^{-1}5 + \cot^{-1}7 + 2\cot^{-1}8$.

20. Discuss the solution of $\cos^2 x - 2m\cos x + 4m^2 + 2m - 1 = 0$ for various values of m.

$\cos^2 x - 2m\cos x + 4m^2 + 2m - 1 = 0$

$(\cos^2 x - 2m\cos x + m^2) + (3m^2 + 2m - 1) = 0$

$(\cos x - m)^2 = -3m^2 - 2m + 1$

$\therefore \cos x = m \pm \sqrt{-3m^2 - 2m + 1}$ ... (1)

(A) $-3m^2 - 2m + 1 \geq 0 \Leftrightarrow 3m^2 + 2m - 1 \leq 0 \Leftrightarrow (3m - 1)(m + 1) \leq 0$

$\Leftrightarrow -1 \leq m \leq \frac{1}{3}$ ... (2)

(B) In order (1) to have solution, $-1 \leq \cos x \leq 1$.

(a) First we consider the root $\cos x = m + \sqrt{-3m^2 - 2m + 1}$

(i) $m + \sqrt{-3m^2 - 2m + 1} \leq 1$

$\Leftrightarrow \sqrt{-3m^2 - 2m + 1} \leq 1 - m$, since by (2), RHS is positive.

$\Leftrightarrow -3m^2 - 2m + 1 \leq (1 - m)^2$

$\Leftrightarrow -3m^2 - 2m + 1 \leq 1 - 2m + m^2$

$\Leftrightarrow 4m^2 \geq 0$, which is always true.

(ii) $-1 \leq m + \sqrt{-3m^2 - 2m + 1}$ is always true since $-1 \leq m$.

(b) Next, we consider the root $\cos x = m - \sqrt{-3m^2 - 2m + 1}$,

(i) $\mathrm{m}-\sqrt{-3\mathrm{m}^2-2\mathrm{m}+1} \le 1$ is always true since $\mathrm{m} \le \frac{1}{3}$

(ii) $-1 \le \mathrm{m}-\sqrt{-3\mathrm{m}^2-2\mathrm{m}+1}$

$\Leftrightarrow \sqrt{-3\mathrm{m}^2-2\mathrm{m}+1} \le \mathrm{m}+1$, both sides are positive by (2)

$\Leftrightarrow -3\mathrm{m}^2-2\mathrm{m}+1 \le (\mathrm{m}+1)^2$

$\Leftrightarrow -3\mathrm{m}^2-2\mathrm{m}+1 \le \mathrm{m}^2+2\mathrm{m}+1$

$\Leftrightarrow 4\mathrm{m}^2+4\mathrm{m} \ge 0$

$\Leftrightarrow \mathrm{m}(\mathrm{m}+1) \ge 0$

$\Leftrightarrow \mathrm{m} \le -1 \text{ or } \mathrm{m} \ge 0$

Joining with (2), $\mathrm{m}=-1 \text{ or } 0 \le \mathrm{m} \le \frac{1}{3}$

In conclusion,

(a) For $\mathrm{m}=-1,\quad \mathrm{m}+\sqrt{-3\mathrm{m}^2-2\mathrm{m}+1}=\mathrm{m}-\sqrt{-3\mathrm{m}^2-2\mathrm{m}+1}$
The given equation is reduced to $\cos^2 x+2\cos x+1=0 \Leftrightarrow (\cos x+1)^2=0$
We have only **one** root, that is, $\cos x=-1$.

(b) For $-1<m<0$, we have **one** roots, $\cos x=\mathrm{m}\pm\sqrt{-3\mathrm{m}^2-2\mathrm{m}+1}$

(c) For $\mathrm{m}=0,\ \cos x=\pm 1$, **two** roots.

(d) For $0<m<\frac{1}{3}$, we have **two** roots, $\cos x=\mathrm{m}\pm\sqrt{-3\mathrm{m}^2-2\mathrm{m}+1}$.

(e) For $\mathrm{m}=\frac{1}{3},\quad -3\mathrm{m}^2-2\mathrm{m}+1=0,\quad \mathrm{m}+\sqrt{-3\mathrm{m}^2-2\mathrm{m}+1}=\mathrm{m}-\sqrt{-3\mathrm{m}^2-2\mathrm{m}+1}$

The given equation is reduced to $9\cos^2 x-6\cos x+1=0 \Leftrightarrow (3\cos x-1)^2=0$

We have only **one** root, that is, $\cos x=\frac{1}{3}$.

If we like to find the roots for x , where $0° \le x < 360°$, then

(a) For $\mathrm{m}=-1,\ x=180°$, **one** root.

(b) For $-1<m<0$, x has **four** roots.

(c) For $\mathrm{m}=0,\ x=0°, 180°$, **two** roots.

(d) For $0<m<\frac{1}{3}$, x has **four** roots.

(e) For $\mathrm{m}=\frac{1}{3}$, x has **two** roots.

21. Show that if $bh\cos\theta+ak\sin\theta=ab$ has roots for $\cos\theta$. They always determine values of θ.

Let $c=\cos\theta$, the given equation becomes

$bhc \pm ak\sqrt{1-c^2}=ab$

$\pm ak\sqrt{1-c^2}=ab-bhc$

$$a^2k^2(1-c^2) = (ab-bhc)^2$$
$$a^2k^2(1-c^2) = a^2b^2 - 2ab^2hc + b^2h^2c^2$$
$$(b^2h^2 + a^2k^2)c^2 - 2ab^2hc + (a^2b^2 - a^2k^2) = 0$$

Since the equation has roots for $c = \cos\theta$,

$$\Delta = (-2ab^2h)^2 - 4(b^2h^2+a^2k^2)(a^2b^2-a^2k^2) = 4\,a^4\,k^4 + 4\,a^2\,b^2\,h^2\,k^2 - 4\,b^2\,k^2\,a^4$$
$$= 4a^2\,k^2\,(b^2\,h^2 + a^2\,k^2 - a^2\,b^2) \geq 0$$
$$a^2\,b^2 \leq b^2\,h^2 + a^2\,k^2 \quad \ldots.(1)$$

Now, $bh\cos\theta + ak\sin\theta = ab$

$$\frac{bh}{\sqrt{b^2h^2+a^2k^2}}\cos\theta + \frac{ak}{\sqrt{b^2h^2+a^2k^2}}\sin\theta = \frac{ab}{\sqrt{b^2h^2+a^2k^2}}$$

Put $\cos\alpha = \frac{bh}{\sqrt{b^2h^2+a^2k^2}}, \sin\alpha = \frac{ak}{\sqrt{b^2h^2+a^2k^2}}$

Then the given equation becomes $\cos\theta\cos\alpha + \sin\theta\sin\alpha = \frac{ab}{\sqrt{b^2h^2+a^2k^2}}$

Or $\cos(\theta-\alpha) = \frac{ab}{\sqrt{b^2h^2+a^2k^2}} \quad \ldots(2)$

(2) has solution $\Leftrightarrow \left|\frac{ab}{\sqrt{b^2h^2+a^2k^2}}\right| \leq 1 \Leftrightarrow \frac{a^2\,b^2}{b^2\,h^2 + a^2\,k^2} \leq 1$, which is always true by (1).

$$\theta - \alpha = 360°n \pm \cos^{-1}\left(\frac{ab}{\sqrt{b^2h^2+a^2k^2}}\right)$$

$\therefore \theta = \alpha + 360°n \pm \cos^{-1}\left(\frac{ab}{\sqrt{b^2h^2+a^2k^2}}\right)$, where $n \epsilon \mathbf{Z}$.

22. Express $\sin\frac{\theta}{2}$ in terms of $\sin\theta$, where θ is in the neighbourhood of $420°$.

For what precise neighbourhood is the result valid ?

$2\sin^2\frac{\theta}{2} = 1 - \cos\theta$, $2\cos^2\frac{\theta}{2} = 1 + \cos\theta$

Replace θ by $(90° - \theta)$,

$2\sin^2\left(45° - \frac{\theta}{2}\right) = 1 - \cos(90° - \theta)$, $2\cos^2\left(45° - \frac{\theta}{2}\right) = 1 + \cos(90° - \theta)$

$\sqrt{2}\sin\left(45° - \frac{\theta}{2}\right) = \pm\sqrt{1 - \sin\theta}$, $\sqrt{2}\cos\left(45° - \frac{\theta}{2}\right) = \pm\sqrt{1 + \sin\theta}$

Since θ is in the neighbourhood of $420°$, $\left(45° - \frac{\theta}{2}\right)$ is in the neighbourhood $-165°$.

Note that $\left(45° - \frac{\theta}{2}\right)$ is in the 3rd quadrant when $270° < \theta < 450°$.

$$\therefore \sqrt{2}\sin\left(45^\circ - \frac{\theta}{2}\right) = -\sqrt{1-\sin\theta}, \quad \sqrt{2}\cos\left(45^\circ - \frac{\theta}{2}\right) = -\sqrt{1+\sin\theta}$$

$$\begin{cases} \sqrt{2}\left(\sin 45^\circ \cos\frac{\theta}{2} - \cos 45^\circ \sin\frac{\theta}{2}\right) = -\sqrt{1-\sin\theta} \\ \sqrt{2}\left(\cos 45^\circ \cos\frac{\theta}{2} + \sin 45^\circ \sin\frac{\theta}{2}\right) = -\sqrt{1+\sin\theta} \end{cases}, \quad 270^\circ < \theta < 450^\circ$$

$$\begin{cases} \cos\frac{\theta}{2} - \sin\frac{\theta}{2} = -\sqrt{1-\sin\theta} & \ldots(1) \\ \cos\frac{\theta}{2} + \sin\frac{\theta}{2} = -\sqrt{1+\sin\theta} & \ldots(2) \end{cases}$$

$$\frac{(2)-(1)}{2}, \quad \sin\frac{\theta}{2} = \frac{1}{2}\left(-\sqrt{1+\sin\theta} + \sqrt{1-\sin\theta}\right), \quad 270^\circ < \theta < 450^\circ.$$

23. Prove that $\tan\frac{\theta}{4}$ is one of the values of $\frac{1\pm\sqrt{1-\sin\theta}}{1\pm\sqrt{1+\sin\theta}}$, and find the other values.

$$\frac{1\pm\sqrt{1-\sin\theta}}{1\pm\sqrt{1+\sin\theta}} = \frac{1\pm\sqrt{\left(\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}\right)-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}}{1\pm\sqrt{\left(\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}\right)+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}} = \frac{1\pm\sqrt{\left(\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right)^2}}{1\pm\sqrt{\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^2}} = \frac{1\pm\left(\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right)}{1\pm\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)}$$

By taking different signs in the fraction, we have:

(a) $$\frac{1+\left(\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right)}{1+\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)} = \frac{1+2\sin\frac{\theta}{4}\cos\frac{\theta}{4}-\left(1-2\sin^2\frac{\theta}{4}\right)}{1+2\sin\frac{\theta}{4}\cos\frac{\theta}{4}+\left(2\cos^2\frac{\theta}{4}-1\right)} = \frac{2\sin\frac{\theta}{4}\cos\frac{\theta}{4}+2\sin^2\frac{\theta}{4}}{2\sin\frac{\theta}{4}\cos\frac{\theta}{4}+2\cos^2\frac{\theta}{4}} = \frac{\sin\frac{\theta}{4}\left(\cos\frac{\theta}{4}+\sin\frac{\theta}{4}\right)}{\cos\frac{\theta}{4}\left(\cos\frac{\theta}{4}+\sin\frac{\theta}{4}\right)} = \frac{\sin\frac{\theta}{4}}{\cos\frac{\theta}{4}} = \mathbf{\tan\frac{\theta}{4}}$$

(b) $$\frac{1-\left(\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right)}{1+\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)} = \frac{1-2\sin\frac{\theta}{4}\cos\frac{\theta}{4}+\left(2\cos^2\frac{\theta}{4}-1\right)}{1+2\sin\frac{\theta}{4}\cos\frac{\theta}{4}+\left(2\cos^2\frac{\theta}{4}-1\right)} = \frac{-2\sin\frac{\theta}{4}\cos\frac{\theta}{4}+2\cos^2\frac{\theta}{4}}{2\sin\frac{\theta}{4}\cos\frac{\theta}{4}+2\cos^2\frac{\theta}{4}} = \frac{\cos\frac{\theta}{4}\left(-\sin\frac{\theta}{4}+\cos\frac{\theta}{4}\right)}{\cos\frac{\theta}{4}\left(\sin\frac{\theta}{4}+\cos\frac{\theta}{4}\right)} = \frac{-\sin\frac{\theta}{4}+\cos\frac{\theta}{4}}{\sin\frac{\theta}{4}+\cos\frac{\theta}{4}}$$

$$= \frac{-\frac{\sin\frac{\theta}{4}}{\cos\frac{\theta}{4}}+1}{\frac{\sin\frac{\theta}{4}}{\cos\frac{\theta}{4}}+1} = \frac{1-\tan\frac{\theta}{4}}{1+\tan\frac{\theta}{4}} = \frac{\tan\frac{\pi}{4}-\tan\frac{\theta}{4}}{1+\tan\frac{\pi}{4}\tan\frac{\theta}{4}} = \tan\left(\frac{\pi}{4}-\frac{\theta}{4}\right) = \mathbf{\tan\left(\frac{\pi-\theta}{4}\right)}$$

(c) $$\frac{1-\left(\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right)}{1-\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)} = \frac{1-2\sin\frac{\theta}{4}\cos\frac{\theta}{4}+\left(2\cos^2\frac{\theta}{4}-1\right)}{1-2\sin\frac{\theta}{4}\cos\frac{\theta}{4}-\left(1-2\sin^2\frac{\theta}{4}\right)} = \frac{2\cos^2\frac{\theta}{4}-2\sin\frac{\theta}{4}\cos\frac{\theta}{4}}{2\sin^2\frac{\theta}{4}-2\sin\frac{\theta}{4}\cos\frac{\theta}{4}} = \frac{\cos\frac{\theta}{4}\left(\cos\frac{\theta}{4}-\sin\frac{\theta}{4}\right)}{-\sin\frac{\theta}{4}\left(\cos\frac{\theta}{4}-\sin\frac{\theta}{4}\right)} = \mathbf{-\cot\frac{\theta}{4}}$$

(d) $$\frac{1+\left(\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right)}{1-\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)} = \frac{1+2\sin\frac{\theta}{4}\cos\frac{\theta}{4}-\left(1-2\sin^2\frac{\theta}{4}\right)}{1-2\sin\frac{\theta}{4}\cos\frac{\theta}{4}-\left(1-2\sin^2\frac{\theta}{4}\right)} = \frac{2\sin^2\frac{\theta}{4}+2\sin\frac{\theta}{4}\cos\frac{\theta}{4}}{2\sin^2\frac{\theta}{4}-2\sin\frac{\theta}{4}\cos\frac{\theta}{4}} = \frac{\sin\frac{\theta}{4}\left(\sin\frac{\theta}{4}+\cos\frac{\theta}{4}\right)}{\sin\frac{\theta}{4}\left(\sin\frac{\theta}{4}-\cos\frac{\theta}{4}\right)} = \frac{\sin\frac{\theta}{4}+\cos\frac{\theta}{4}}{\sin\frac{\theta}{4}-\cos\frac{\theta}{4}}$$

$$= \frac{\frac{\sin\frac{\theta}{4}}{\cos\frac{\theta}{4}}+1}{\frac{\sin\frac{\theta}{4}}{\cos\frac{\theta}{4}}-1} = -\frac{1+\tan\frac{\theta}{4}}{1-\tan\frac{\theta}{4}} = -\frac{1+\tan\frac{\pi}{4}\tan\frac{\theta}{4}}{\tan\frac{\pi}{4}-\tan\frac{\theta}{4}} = -\frac{1}{\tan\left(\frac{\pi}{4}-\frac{\theta}{4}\right)} = \mathbf{-\cot\left(\frac{\pi-\theta}{4}\right)}$$

24. Prove that $\cos\frac{\theta}{2} = (-1)^{\left[\frac{\theta+\pi}{2\pi}\right]}\sqrt{\frac{1}{2}(1+\cos\theta)}$, [x]=greatest integer smaller or equal to x

$\cos 2\theta = 2\cos^2\theta - 1$

Replace θ by $\frac{\theta}{2}$ and rearrange we get $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1}{2}(1+\cos\theta)}$

The point is to determine the $\pm$ sign exactly.

Now, $\cos\alpha$ is positive when α is in the 1st and 4th quadrants, and negative in the 2nd and 3rd quadrants. The sign of $\cos\alpha$ is $(-1)^{\left[\frac{\alpha+\frac{\pi}{2}}{\pi}\right]}$. Here in $\left[\frac{\alpha+\frac{\pi}{2}}{\pi}\right]$, the change from positive to negative is in every turn of π and we add $\frac{\pi}{2}$ since (-1) to power an even number is 1.

Thus the sign for $\cos\frac{\theta}{2}$ should be $(-1)^{\left[\frac{\frac{\theta}{2}+\frac{\pi}{2}}{\pi}\right]} = (-1)^{\left[\frac{\theta+\pi}{2\pi}\right]}$.

$\therefore \cos\frac{\theta}{2} = (-1)^{\left[\frac{\theta+\pi}{2\pi}\right]}\sqrt{\frac{1}{2}(1+\cos\theta)}$

25. If p is an integer and $-1 < q < 1$, find the number of possible values of $\sin x$, such that
(i) $\sin 2px = q$, **(ii)** $\sin(2p+1)x = q$.

(i) The general solution for $\sin 2px = q$ is $2px = n\pi + (-1)^n \sin^{-1} q$, where $n \epsilon \mathbf{Z}$.

Hence $x = \frac{1}{2p}[n\pi + (-1)^n \sin^{-1} q] = \frac{n\pi}{2p} + \frac{1}{2p}(-1)^n \sin^{-1} q$.

If we take $0 \le x < 2\pi$, there are $4p$ values of x, that is, when $n = 0,1,2,\ldots,(4p-1)$

Thus there are $\mathbf{4p}$ values of $\sin x = \sin\left[\frac{n\pi}{2p} + \frac{1}{2p}(-1)^n \sin^{-1} q\right]$.

(ii) The general solution for $\sin(2p+1)x = q$ is
$(2p+1)x = n\pi + (-1)^n \sin^{-1} q$, where $n \epsilon \mathbf{Z}$.

Hence $x = \frac{1}{2p+1}[n\pi + (-1)^n \sin^{-1} q] = \frac{n\pi}{2p+1} + \frac{1}{2p+1}(-1)^n \sin^{-1} q$.

If we take $0 \le x < 2\pi$, there are $2(2p+1) = 4p+2$ values of x, that is, when $n = 0,1,2,\ldots,(4p+1)$.

Since $\sin x = \sin\left[\frac{n\pi}{2p+1} + \frac{1}{2p+1}(-1)^n \sin^{-1} q\right]$, and amongst these solutions

$\sin\left[\frac{k\pi}{2p+1} + \frac{1}{2p+1}(-1)^k \sin^{-1} q\right] = \sin\left[\frac{[(2p+1)+k]\pi}{2p+1} + \frac{1}{2p+1}(-1)^{[(2p+1)+k]} \sin^{-1} q\right]$

for all $k = 0,1,\ldots,(2p-2)$.

Thus there are only $\mathbf{2p-1}$ values of $\sin x$.

(For those who do not understand the proof may start with solving $\sin 2x = 1, \sin 3x = 1, \ldots$ and then find the possible values of $\sin x$.)

26. Solve $x^5 - 5k^2x^3 + 5k^4x = 2k^5 \cos\alpha$, for x in terms of α and k .

Put $x = 2k\cos\theta$, then the given equation becomes:

$(2k\cos\theta)^5 - 5k^2(2k\cos\theta)^3 + 5k^4(2k\cos\theta) = 2k^5\cos\alpha$

$32k^5\cos^5\theta - 40k^5\cos^3\theta + 10k^5\cos\theta = 2k^5\cos\alpha$

$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = \cos\alpha$

$\cos 5\theta = \cos\alpha$

(The proof that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ is left for the reader.)

$5\theta = 2n\pi \pm \alpha$

$\therefore \theta = \frac{2n\pi\pm\alpha}{5}, \ n\epsilon Z$

27. Simplify $\tan^{-1}\frac{p-q}{1+pq} + \tan^{-1}\frac{q-r}{1+qr}$.

Take $p = \tan x, q = \tan y, r = \tan z$.

$\tan^{-1}\frac{p-q}{1+pq} + \tan^{-1}\frac{q-r}{1+qr} = \tan^{-1}\frac{\tan x-\tan y}{1+\tan x\tan y} + \tan^{-1}\frac{\tan y-\tan z}{1+\tan y\tan z}$

$= \tan^{-1}[\tan(x-y)] + \tan^{-1}[\tan(y-z)] = (x-y) + (y-z) = x - z = \underline{\underline{\mathbf{\tan^{-1}p - \tan^{-1}r}}}$

28. Prove that $\tan^{-1}\frac{1}{p} = \tan^{-1}\frac{1}{p+q} + \tan^{-1}\frac{q}{p^2+pq+1}$.

$\tan\left[\tan^{-1}\frac{1}{p+q} + \tan^{-1}\frac{q}{p^2+pq+1}\right] = \frac{\frac{1}{p+q}+\frac{q}{p^2+pq+1}}{1-\frac{1}{p+q}\times\frac{q}{p^2+pq+1}} = \frac{(p^2+pq+1)+q(p+q)}{(p+q)(p^2+pq+1)-q} = \frac{p^2+2\,p\,q+q^2+1}{p\,(p^2+2\,p\,q+q^2+1)} = \frac{1}{p}$

$\therefore \tan^{-1}\frac{1}{p} = \tan^{-1}\frac{1}{p+q} + \tan^{-1}\frac{q}{p^2+pq+1}$.

29. Use the result of No.28 to express $\frac{\pi}{4}$ in the form of $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$.

Also express $\tan^{-1}\frac{1}{2}$ and $\tan^{-1}\frac{1}{3}$ each in the form of $\tan^{-1}\frac{1}{m} + \tan^{-1}\frac{1}{n}$ where m and n are positive integers.

Put $p = q = 1$, then $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \tan^{-1}1 = \frac{\pi}{4}$.

Put $p = 2, q = 1$, then $\tan^{-1}\frac{1}{2} = \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$

Put $p = 3, q = 1$, then $\tan^{-1}\frac{1}{3} = \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{13}$

30. Prove that $\frac{\pi}{4} = 2\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{13}$.

$$\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \left(\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}\right) + \tan^{-1}\frac{1}{3} = 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$$

$$= 2\left(\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{13}\right) + \tan^{-1}\frac{1}{7} = 2\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{13}.$$

31. Prove that $\frac{\pi}{4} = 2\cot^{-1}5 + \cot^{-1}7 + 2\cot^{-1}8$.

From No. 27, $\tan^{-1}\frac{1}{p} = \tan^{-1}\frac{1}{p+q} + \tan^{-1}\frac{q}{p^2+pq+1}$

Since $\tan^{-1}\frac{1}{p} = \cot^{-1}p$, (Proof is left to the reader.)

We have $\cot^{-1}p = \cot^{-1}(p+q) + \cot^{-1}\frac{p^2+pq+1}{q}$

Put $p = q = 1$, then $\frac{\pi}{4} = \cot^{-1}1 = \cot^{-1}2 + \cot^{-1}3$

Put $p = 2, q = 1$, then $\cot^{-1}2 = \cot^{-1}3 + \cot^{-1}7$

Put $p = 3, q = 2$, then $\cot^{-1}3 = \cot^{-1}5 + \cot^{-1}8$

Combining, $\frac{\pi}{4} = \cot^{-1}2 + \cot^{-1}3 = (\cot^{-1}3 + \cot^{-1}7) + \cot^{-1}3$

$$= 2\cot^{-1}3 + \cot^{-1}7 = 2(\cot^{-1}5 + \cot^{-1}8) + \cot^{-1}7$$
$$= 2\cot^{-1}5 + \cot^{-1}7 + 2\cot^{-1}8$$

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